

Entanglement evolution of two qubits under noisy environments

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The entanglement evolution of two qubits under local, single- and two- sided noisy channels is investigated. It is found that for all pure initial states, the entanglement under a one-sided noisy channel is completely determined by the maximal trace distance which is the main element to construct the measure of non-Markovianity. For the two-sided noisy channel case, when the qubits are initially prepared in a general class of states, no matter pure or mixed, the entanglement can be expressed as the products of initial entanglement and the channels' action on the maximally entangled state.

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I. INTRODUCTION

In realistic quantum information processing [1, 2], quantum systems are open to essentially uncontrollable environments that act as sources of decoherence and dissipation[3, 4]. Therefore the study of entanglement evolution of quantum systems taking into account the effect of environment has become more and more interesting[5–10]. In Ref.[8] Konrad *et al.* provided a direct relationship between the initial and final entanglement of two qubits with one qubit subject to incoherent dynamics. It has been shown that given any one-sided quantum channel, the concurrence of output state corresponding to any initial pure input state of interest can always be equivalently obtained by the product of the concurrence of input state and that of the output state with the maximally entangled state as an input state. However, for the two-sided quantum channel case or the mixed initial state case, the product of the two concurrences only provides an upper bound for the concurrence of interest. It is obviously important to find the exact relations of that for two-sided noisy channel case and that for the mixed initial state case.

Over the past decades, the conventionally employed Markovian approximation with the assumption of an infinitely short correlation time of environment has experienced more and more challenges due to the advance of experimental techniques[3]. Recent studies [11–25] have shown that non-Markovian quantum processes play an increasingly important role in many fields of physics. In order to quantitatively study the non-Markovian dynamics, following the first computable measure of Markovianity for quantum channels introduced in Ref.[26], some measures for the degree of non-Markovianity have been introduced[27, 28]. It is very interesting to investigate the explicit relationship between the entanglement evolution

and the non-Markovianity.

In this paper, we consider the entanglement evolution of two qubits under local noisy channels. The main aim of this work is to analyze if and to what extent can the factorization law given in Ref.[8] be generalized. As a central result, we demonstrate that for the two-sided noisy channels, even when the qubits are initially prepared in some mixed states, the concurrence can be expressed as the products of the initial concurrence and the channels' action on the maximally entangled state.

The paper is organized as follows: In Sec.II, we present the model and its analytical solution. In Sec.III we give the entanglement dynamics of two qubits under local, single- and two- sided non-Markovian channels. In Sec.IV two applications are given. Finally we give a conclusion of our results in Sec.V.

II. THE MODEL

We consider a system formed by two noninteracting parts A and B , each part consisting of a qubit $s = a, b$ locally interacting respectively with a reservoir $R_S = R_A, R_B$. Each qubit and the corresponding reservoir are initially considered independent. The dynamics of each part, consisting a qubit with excited state $|e\rangle$ and ground state $|g\rangle$ which is coupled to a reservoir of field modes initially in the vacuum state, can be represented by the reduced density matrix[3]

$$\rho^S(t) = \begin{pmatrix} \rho_{ee}^S(0)|h_S(t)|^2 & \rho_{eg}^S(0)h_S(t) \\ \rho_{ge}^S(0)h_S^*(t) & 1 - \rho_{ee}^S(0)|h_S(t)|^2 \end{pmatrix} \quad (1)$$

in the qubit basis $\{|e\rangle, |g\rangle\}$, where the superscript $S = A, B$ represents part S . The function $h_S(t)$ ($h(t)$ for short) is defined as the solution of the integrodifferential equation

$$\frac{d}{dt}h(t) = - \int_0^t dt_1 f(t-t_1)h(t), \quad (2)$$

with $f(t-t_1)$ denotes the two-point reservoir correlation function which can be written as the Fourier Transform

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of the spectral density $J(\omega)$:

$$f(t - t_1) = \int d\omega J(\omega) \exp[i(\omega_0 - \omega)(t - t_1)]. \quad (3)$$

The exact form of $h(t)$ thus depends on the particular choice of the spectral density of the reservoir. It should be noted that the dynamical map of the model can be Markovian (iff $|h(t)|$ is a monotonically decreasing function of time) and non-Markovian (iff $|h(t)|$ increases at some times)[29].

Next, we are ready to use the reduced density matrix elements given in Eq.(1) to construct the reduced density matrix for the two-qubit system. Following the procedure presented in Ref.[25], we find that in the standard product basis $B = \{|1\rangle = |ee\rangle, |2\rangle = |eg\rangle, |3\rangle = |ge\rangle, |4\rangle = |gg\rangle\}$, the diagonal elements of the reduced density matrix $\rho(t)$ for the two-qubits system can be written as

$$\begin{aligned} \rho_{11}(t) &= \rho_{11}(0)|h_A(t)|^2|h_B(t)|^2, \\ \rho_{22}(t) &= |h_A(t)|^2[\rho_{22}(0) + \rho_{11}(0)(1 - |h_B(t)|^2)], \\ \rho_{33}(t) &= |h_B(t)|^2[\rho_{33}(0) + \rho_{11}(0)(1 - |h_A(t)|^2)], \\ \rho_{44}(t) &= 1 - (\rho_{11}(t) + \rho_{22}(t) + \rho_{33}(t)), \end{aligned} \quad (4)$$

and the nondiagonal elements are

$$\begin{aligned} \rho_{12}(t) &= \rho_{12}(0)|h_A(t)|^2h_B(t), \\ \rho_{13}(t) &= \rho_{13}(0)h_A(t)|h_B(t)|^2, \\ \rho_{14}(t) &= \rho_{14}(0)h_A(t)h_B(t), \\ \rho_{23}(t) &= \rho_{23}(0)h_A(t)h_B(t), \\ \rho_{24}(t) &= h_A(t)[\rho_{24}(0) + \rho_{13}(0)(1 - |h_B(t)|^2)], \\ \rho_{34}(t) &= h_B(t)[\rho_{34}(0) + \rho_{12}(0)(1 - |h_A(t)|^2)], \end{aligned} \quad (5)$$

and $\rho_{ij}(t) = \rho_{ji}^*(t)$. We should note that if we let $h_A(t) = h(t)$ and $h_B(t) = 1$, this reduced density matrix represents the dynamics of the two qubits in the single-sided channel case.

III. ENTANGLEMENT EVOLUTION

In this section, we give the entanglement evolution for single-sided channel case ($h_A(t) = h(t)$, $h_B(t) = 1$) and two-sided channels case for pure and mixed initial states. Firstly, we assume that the qubits are initially prepared in the pure state

$$|\Psi\rangle = c_1|ee\rangle + c_2|eg\rangle + c_3|ge\rangle + c_4|gg\rangle, \quad (6)$$

where $c_i (i = 1, 2, 3, 4)$ are complex and satisfy the normalization condition $\sum_{i=1}^4 |c_i|^2 = 1$. The elements of the density matrix corresponding to this initial state can be written as $\rho_{ij}(0) = c_i c_j^*$, ($i, j = 1, 2, 3, 4$). In what follows, we consider the single-sided channel case. Substituting $h_A(t) = h(t)$, $h_B(t) = 1$ and $\rho_{ij}(0) = c_i c_j^*$ into Eqs. (4) and (5) we obtain the diagonal elements

$$\begin{aligned} \rho_{11}(t) &= c_1^* c_1 |h(t)|^2, \\ \rho_{22}(t) &= c_2^* c_2 |h(t)|^2, \\ \rho_{33}(t) &= c_3^* c_3 + c_1^* c_1 (1 - |h(t)|^2), \\ \rho_{44}(t) &= c_4^* c_4 + c_2^* c_2 (1 - |h(t)|^2), \end{aligned} \quad (7)$$

and the nondiagonal elements

$$\begin{aligned} \rho_{12}(t) &= c_1 c_2^* |h(t)|^2, \\ \rho_{13}(t) &= c_1 c_3^* h(t), \\ \rho_{14}(t) &= c_1 c_4^* h(t), \\ \rho_{23}(t) &= c_2 c_3^* h(t), \\ \rho_{24}(t) &= c_2 c_4^* h(t), \\ \rho_{34}(t) &= c_3 c_4^* + c_1 c_2^* (1 - |h(t)|^2), \end{aligned} \quad (8)$$

and $\rho_{ij}(t) = \rho_{ji}^*(t)$.

To quantify the entanglement we use Wootters concurrence[30]. From Eqs. (7) and (8), we find that the concurrence of the two qubits is

$$C_\Psi(t) = C_\Psi(0)|h(t)|, \quad (9)$$

where $C_\Psi(0) = 2|c_1 c_4 - c_2 c_3|$ is the initial entanglement of state (6). Equation (9) shows that the entanglement can be factorized as two terms whose physical meaning is given as follows. We consider the extreme case in which the two qubits are initially prepared in the maximally entangled state, i.e., $C_\Psi(0) = 1$. Then equation (9) reduces to $C_\Psi(t) = |h(t)|$, that means that $|h(t)|$ stands for the time evolution of the entanglement for the maximally entangled initial state under the single-sided channel. Then we can say that the entanglement reduction under single-sided noisy channel is equal to the product of the initial entanglement $C_\Psi(0)$ and the time evolution of the entanglement for the maximally entangled initial state. That is, the dynamics of the entanglement is completely determined by the time evolution of the entanglement for the initial state with maximal entanglement.

The factorization law given in Eq.(9) is valid for single-sided channel case. One may ask to what extent it can be generalized to the two-sided channels case. In what follows, we will show that this can be done by constraining the initial state to a certain class of states. We also let the two qubits be initially prepared in the pure state (6). Substituting $\rho_{ij}(0) = c_i c_j^*$ into Eqs. (4) and (5) and after some calculations we find that the concurrence at time t is given by

$$C_\Psi(t) = \max\{0, Q(t)\}. \quad (10)$$

The explicit expression of $Q(t)$ is

$$Q(t) = C_\Psi(0)|h_A(t)||h_B(t)|\mathcal{X}, \quad (11)$$

where $\mathcal{X} = 1 - |c_1|^2/|c_2 c_3 - c_1 c_4|\sqrt{\xi}$ with $\xi = (1 - |h_A(t)|^2)(1 - |h_B(t)|^2)$. Equation (10) shows that for the case of two-sided noisy channels, the entanglement evolution is not equal to the product of the initial entanglement $C_\Psi(0)$ and the maximal entanglement evolution of single-sided channel $|h_A(t)|$ and $|h_B(t)|$ but an additional term \mathcal{X} should be taken into account. We should note that if for some time intervals $\mathcal{X} < 0$, the entanglement sudden death occurs. Interestingly if we consider the case $c_1 = 0$, that is, the two qubits are initially prepared in the state

$$|\Phi\rangle = c_2|eg\rangle + c_3|ge\rangle + c_4|gg\rangle, \quad (12)$$

with $\sum_{i=2}^4 |c_i|^2 = 1$, we find that $\mathcal{X} = 1$ and Eq.(10) reduces to

$$C_\Phi(t) = C_\Phi(0)|h_A(t)||h_B(t)|, \quad (13)$$

with $C_\Phi(0) = 2|c_2c_3|$ being the entanglement of initial state (12). Equation (13) indicates that for the pure initial state (12), the entanglement evolution under two-sided noisy channels is equal to the product of the initial entanglement $C_\Phi(0)$, $|h_A(t)|$ (the maximal entanglement evolution under single-sided channel A) and $|h_B(t)|$ (the maximal entanglement evolution under single-sided channel B).

As we know, quantum mechanics and quantum information processing are not constrained to pure states. So we give the generalization of the factorization law to the mixed state. We consider the case in which the qubits are initially prepared in the mixed state

$$\rho(0) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & b & z & e \\ 0 & z^* & c & f \\ 0 & e^* & f^* & d \end{pmatrix} \quad (14)$$

where b, c and d are real numbers and satisfy $b + c + d = 1$. For this state it is easy to check that the initial concurrence $C_\rho(0) = \max\{0, \sqrt{bc} + |z| - |\sqrt{bc} - |z||\}$ and the terms $\rho_{ij}(0)$ in Eqs.(4) and (5) can be written as

$$\begin{aligned} \rho_{11}(0) &= 0, & \rho_{12}(0) &= 0, & \rho_{13}(0) &= 0, & \rho_{14}(0) &= 0, \\ \rho_{21}(0) &= 0, & \rho_{22}(0) &= b, & \rho_{23}(0) &= z, & \rho_{24}(0) &= e, \\ \rho_{31}(0) &= 0, & \rho_{32}(0) &= z^*, & \rho_{33}(0) &= c, & \rho_{34}(0) &= f, \\ \rho_{41}(0) &= 0, & \rho_{42}(0) &= e^*, & \rho_{43}(0) &= f^*, & \rho_{44}(0) &= d. \end{aligned}$$

After some calculations we find that the concurrence at time t is given by

$$C_\rho(t) = C_\rho(0)|h_A(t)||h_B(t)|. \quad (15)$$

That is the factorization law for the two-sided noisy channel case. We should note that the initial states (12) and (14) have a common property, that is, the number of excitations in the system is not more than one. We denote these states NOE states. Then we can conclude that for the two-sided noisy channels case, when the qubits are initially prepared in NOE states, no matter pure or mixed, the concurrence can be expressed as the products of initial concurrence and the channels' action on the maximally entangled state.

IV. APPLICATIONS

In this section we give two applications of our results. The first one is the entanglement preservation. Equations (10) and (15) show the direction for entanglement preservation, that is, choosing appropriate parameters to make the values of $|h_A(t)|$ and $|h_B(t)|$ around 1. As an

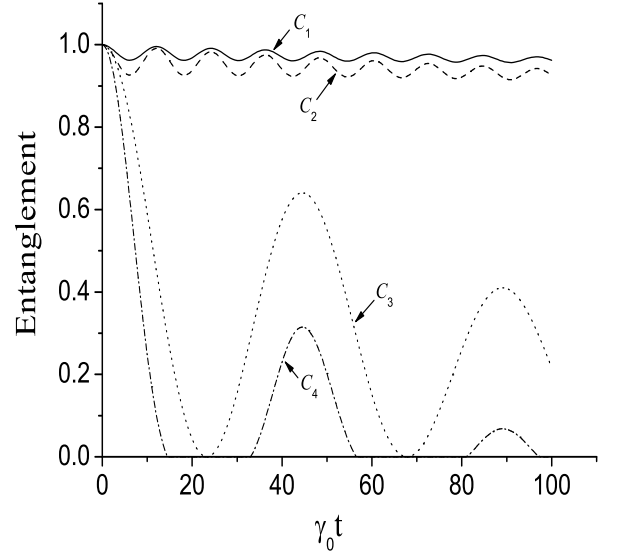


FIG. 1: Dynamics of entanglement as functions of $\gamma_0 t$.

example, we investigate the detuning case of a Lorentian spectral density[3]

$$J(\omega) = \frac{1}{2\pi} \frac{\gamma_0 \lambda^2}{(\omega_0 - \omega - \Delta)^2 + \lambda^2}, \quad (16)$$

where $\Delta = \omega_0 - \omega_c$ is the detuning of ω_c and ω_0 , and ω_c is the center frequency of the cavity. λ defines the spectral width of the reservoir and is connected to the reservoir correlation time $\tau_R = \lambda^{-1}$. γ_0 is related to the decay of the excited state of the qubit in the Markovian limit of a flat spectrum, the relaxation time scale τ_S over which the state of the system changes is then related to γ_0 by $\tau_S = \gamma_0^{-1}$. We evaluate the reservoir correlation function $f(t - t_1)$ using the spectral density $J(\omega)$ and obtain

$$f(t - t_1) = \frac{1}{2} \gamma_0 \lambda \exp[-(\lambda - i\Delta)(t - t_1)]. \quad (17)$$

Solving equation (2) with this correlation function, we find

$$h(t) = e^{-\frac{1}{2}(\lambda - i\Delta)t} \left[\cosh\left(\frac{dt}{2}\right) + \frac{\lambda - i\Delta}{d} \sinh\left(\frac{dt}{2}\right) \right] \quad (18)$$

with $d = \sqrt{(\lambda - i\Delta)^2 - 2\gamma_0\lambda}$. From Eq.(18) we find that the value of $|h(t)|$ is determined by the parameters Δ and λ . For the two-sided channel, we denote Δ_A and λ_A for channel A and Δ_B and λ_B for channel B .

Figure 1 shows the dynamics of the entanglement as functions of $\gamma_0 t$ for different cases. In Fig.1, C_1 is the entanglement evolution of the initial state $\psi = 1/\sqrt{2}(|eg\rangle + |ge\rangle)$ for the single-sided channel case and C_2 is that for the two-sided channel case. We choose the parameters $\Delta_A = \Delta_B = 0.5\gamma_0$ and $\lambda_A = \lambda_B = 0.01\gamma_0$ to make the values of $|h_A(t)|$ and $|h_B(t)|$ around 1. Clearly, the entanglement is protected well.

In Fig.1, we also give the entanglement evolutions for two-sided channel case of initial state $\psi = 1/\sqrt{2}(|eg\rangle +$

$|ge\rangle\rangle$ (C_3) and initial state $\phi = 1/\sqrt{2}(|ee\rangle + |gg\rangle)$ (C_4). Here, we choose the parameters $\Delta_A = \Delta_B = 0$ and $\lambda_A = \lambda_B = 0.01\gamma_0$. Comparing C_3 and C_4 we find that for initial state ψ (NOE state), the entanglement is always large than zero except some discrete time instants. This can be understood as follows. For the NOE states the factorization law (13) is valid. The entanglement is determined by $|h_A(t)|$ and $|h_B(t)|$ of which the values are not less than zero for all the parameters. However, for the initial state ϕ (not NOE state), the entanglement sudden death and the entanglement sudden birth can occur. This can be understood from Eqs.(10) and (11) that in this case \mathcal{X} can be less than zero for some time intervals.

The second application is to show the connection of the entanglement and the non-Markovianity of the channels. The measure for non-Markovianity defined in Ref. [27] is

$$\mathcal{N} = \max_{\rho_{1,2}(0)} \int_{\sigma>0} dt \sigma(t), \quad (19)$$

where $\sigma(t)$ is the rate of change of the trace distance which can be defined as

$$\sigma(t) = \frac{d}{dt} D(\rho_1(t), \rho_2(t)), \quad (20)$$

with $D(\rho_1, \rho_2) = \frac{1}{2}|\rho_1 - \rho_2|$ being the trace distance of the quantum states ρ_1 and ρ_2 . Here, $D(\rho_1, \rho_2)$ describes the distinguishability between the two states and satisfying $0 \leq D \leq 1$. We should note that if there exists a pair of initial states and a certain time t such that $\sigma(t) > 0$, the process is non-Markovian. Physically, this means that for non-Markovian dynamics the distinguishability of the pair of states increases at certain times. This can be interpreted as a flow of information from the environment back to the system which enhances the possibility of distinguishing the two states. Following Ref.[27], reference [31] presented a practical idea for directly measuring the non-Markovian character of a single qubit coupled to a zero-temperature bosonic reservoir. It is easy to check that any pair of initial states satisfying the conditions in the theorem given in Ref.[31] definitely owns the same maximal trace distance

$$D(\rho_1(t), \rho_2(t)) = |h(t)|, \quad (21)$$

namely, $|h(t)|$ stands for the maximal trace distance of one part (A or B) of our model. From Eq.(9) we can

find that the entanglement evolution under single-sided noisy channel is equal to the product of the initial entanglement $C_\Psi(0)$ and the maximal trace distance $|h(t)|$. We also know that $|h(t)|$ also stands for the channels' action on the maximally entangled state. It is evident from this relation that any increase of the entanglement implies occurring of the non-Markovian dynamics. Hence a conceptually simple way to quantify the degree of non-Markovianity of an unknown quantum evolution would be to compute the amount of entanglement at different times within a selected interval and check for strict monotonic decrease of the quantum correlations. This measurement has been given in the very recent paper[28].

V. CONCLUSION

In summary, we have studied the entanglement dynamics of two qubits under local, single- and two-sided noisy channels. We find that in the case where the initial state is pure, and only one of the subsystems undergoes a non-Markovian channel, the equation of motion for entanglement presents the form of a simple factorization law –the second term contains only information about how the maximal entanglement is affected by the dynamics, and the first term scales the second by the initial amount of entanglement. For more realistic scenarios, where both parts are influenced by the local environments, when the qubits are initially prepared in NOE states, no matter pure or mixed, the concurrence can be expressed as the products of initial concurrence and the channels' action on the maximally entangled state. Using these factorization laws we have found the way to protect entanglement from decay and given a connection of the entanglement and the non-Markovianity of the channels.

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